Fall 2018

MATH3060 Mathematical Analysis III

Selected Solutions to Mid-Term Examination

Part A (50 marks) Each question carries 10 marks.

1. Find the Fourier series of the function $f(x) = \sin x,\ x \in [0, \pi]$, and $f(x) = 0,\ x \in [-\pi, 0)$. Does the Fourier series of f converge to f uniformly?

2. Let g be a 2π-periodic function satisfying the Lipschitz condition, that is, for some constant L, $|g(x) - g(y)| \leq L|x - y|$ for all x, y. Show that its Fourier coefficients satisfies
$$|c_n| \leq \frac{C}{n}, \quad n \in \mathbb{Z},$$
for some constant C. Express C in terms of L.

3. Let $S_n\varphi(x)$ be the n-th partial sum of the Fourier series of the function φ defined by
$$\varphi(x) = \frac{\sin 2x}{x}, \quad x \in (0, \pi],$$
and $\varphi(x) = -6,\ x \in [-\pi, 0)$. Find $\lim_{n\to\infty} S_n\varphi(0)$. You should justify your answer.

4. Consider $(\mathbb{R}, d)$ where d is the discrete metric d. Show that every real-valued function in X is continuous with respect to d. Can you find a metric d' on $\mathbb{R}$ such that the only continuous real-valued functions on $(\mathbb{R}, d')$ are the constant ones?

   Solution 1. Let $x_n \to x$. We claim $f(x_n) \to f(x)$. As $x_n \to x$, for $\varepsilon > 0$, there is some n_0 such that $d(x_n, x) < \varepsilon$ for all $n \geq n_0$. But when $\varepsilon = 1/2$, $d(x_n, x) < 1/2$ means $x_n = x$ for all n_0 (the only corresponding to $1/2$). Therefore, $f(x_n) = f(x)$ for all $n \geq n_0$, so trivially $f(x_n) \to f(x)$. We conclude that f is continuous.

Solution 2. Use the fact $B_{1/2}(x) \subset \{x\}$ which implies that $\{x\}$ is an open set. Since the union of any open sets is still an open set, every subset of X is open. Now, a function is continuous if and only if $f^{-1}(G)$ for any open set G in $\mathbb{R}$. As $f^{-1}(G)$ is a subset of X and hence it must be open, f is continuous.

5. State Minkowski's Inequality for functions in $C[a,b]$ and then deduce it from Hölder's Inequality:

$$\int_a^b |fg| \leq \left(\int_a^b |f|^p\right)^{1/p} \left(\int_a^b |g|^q\right)^{1/q}, \quad \frac{1}{p} + \frac{1}{q} = 1, \ p > 1 \ .$$

Part B (50 marks)

(6) Consider the norms $\|\cdot\|_1$ and $\|\cdot\|_p, 1 < p < \infty$, on $C[a,b]$ and denote their respective induced metrics by d_1 and d_p.

(a) (10 marks) Show that d_1 is weaker than d_p.

(a) (10 marks) Show that d_1 is strictly weaker than d_p.

(b) (5 marks) Is (a) still valid on $C(-\infty, \infty)$? (Improper integrals involved.)

Solution. (c). No longer valid. Consider a positive, continuous function which is even and is equal to $1/x$ for $x > 1$. Because

$$\int_{-\infty}^{\infty} f \geq \int_1^{\infty} \equiv \lim_{r \to \infty} \int_1^r \frac{1}{x}\, dx = \lim_{r \to \infty} \log r = \infty \ ,$$

the improper integral of f does not exist, that is, $\|f\|_1 = \infty$. On the other hand, using

$$\int_1^r f^p(x)\, dx = \int_1^r x^{-p}\, dx = \frac{1}{p-1}(1 - r^{1-p}) < \frac{1}{p-1} \ ,$$

we see that $\|f\|_p$ is finite.

(7) Let f be a 2π-periodic (real-valued) integrable function in $[-\pi, \pi]$ and denote the n-th partial sum of its Fourier series by $S_n f$. Set

$$\sigma_n f(x) = \frac{1}{n} \sum_{k=0}^{n-1} S_k f(x) \ .$$

(a) (10 marks) Show that

$$\sigma_n f(x) = \frac{1}{2\pi n} \int_{-\pi}^{\pi} \frac{\sin^2(nz/2)}{\sin^2(z/2)} f(x+z)\, dz \ , n \geq 1 \ .$$

(b) (5 marks) Show that

$$1 = \frac{1}{2\pi n} \int_{-\pi}^{\pi} \frac{\sin^2(nz/2)}{\sin^2(z/2)}\, dz \ .$$

(c) (10 marks) Prove that when f is continuous at x,

$$\lim_{n\to\infty} \sigma_n f(x) = f(x).$$

Solution. (a) It is based on the summation formula:

$$\sum_{k=0}^{n-1} \sin(k+\frac{1}{2})x = \frac{\sin^2(nx/2)}{\sin^2(x/2)} \ .$$

To prove it, we observe that

$$2 \sin\frac{x}{2} \sin(k+\frac{1}{2})x = \cos kx - \cos(k+1)x \ ,$$

and, by adding up, we get the formula after using $1 - \cos nx = 2\sin^2(nx/2)$.

(c) Consider

$$\begin{aligned}
|\sigma_n f(x) - f(x)| &= \left| \frac{1}{2n\pi} \int_{-\pi}^{\pi} \frac{\sin^2(nz/2)}{\sin^2 z/2} (f(x+z) - f(x))\, dz \right| \\
&\leq \left| \frac{1}{2n\pi} \int_{-\delta}^{\delta} \frac{\sin^2(nz/2)}{\sin^2 z/2} (f(x+z) - f(x))\, dz \right| \\
&\quad + \left| \frac{1}{2n\pi} \left(\int_{-\pi}^{-\delta} + \int_{\delta}^{\pi} \right) \frac{\sin^2(nz/2)}{\sin^2 z/2} (f(x+z) - f(x))\, dz \right| \\
&\equiv I + II \ .
\end{aligned}$$

Let $\varepsilon > 0$, by the continuity of f there is some δ such that $|f(z+x) - f(x)| < \dfrac{\varepsilon}{2}$ for $z, |z| < \delta$. Then,

$$\begin{aligned} I &\leq \frac{1}{2n\pi} \int_{-\delta}^{\delta} \frac{\sin^2(nz/2)}{\sin^2 z/2} |f(x+z) - f(x)|, dz \\ &\leq \frac{\varepsilon}{2} \frac{1}{2n\pi} \int_{-\delta}^{\delta} \frac{\sin^2(nz/2)}{\sin^2 z/2}\, dz \\ &\leq \frac{\varepsilon}{2} \frac{1}{2n\pi} \int_{-\pi}^{\pi} \frac{\sin^2(nz/2)}{\sin^2 z/2}\, dz \\ &= \frac{\varepsilon}{2}\,. \end{aligned}$$

Note that we have use the positivity of the kernel and (b). Now, II goes to zero as $n \to \infty$ by Riemann-Lebesgue Lemma and that is it.